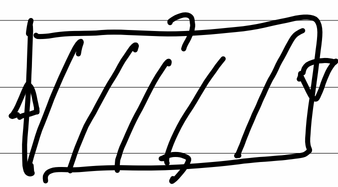
Lemma 2.13

A pair $\alpha^p\beta^q$, $\alpha^r\beta^s \in \pi_1(K)$ of non-trivial elements is represented up to conjugacy by a pair w_1, w_2 of simple loops meeting transversely in a single point, with w_1 orientation preserving and

w_2 orientation reversing iff $p = \pm 1$, $q = 0$ and $s = \pm 1$.

<u>Proof</u>

If $p = \pm 1$, $s = \pm 1$, $q = 0$ then we take $w_1 = z^{\pm 1}$, and w_2 given by

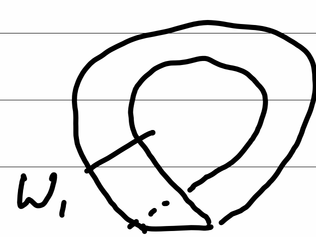


Now suppose we have w_1, w_2 as described. $\exists$ a n'hood N of w_2 which is a Möbius band, and

$\overline{K \setminus N}$ is also a Möbius band:

Indeed, the preimage of N in the oriented double cover T^2 of K must be an annulus, and hence the preimage of $\overline{K \setminus N}$ is also an annulus. But $\partial \overline{K \setminus N}$ has one connected component, so it has to be a Möbius band.

Now w_1 cuts both of these into squares



.

Hence, there is a homeomorphism $h: K \to K$ taking α to ω_1 and β to ω_2.

Hence ω_1 represents the conjugacy class of $h_*(\alpha) = \alpha^p \beta^q$ and ω_2 of $h_*(\beta) = \alpha^r \beta^s$.

Hence q is even and s is odd, since $\alpha \in w(K)$, $\beta \notin w(K)$.

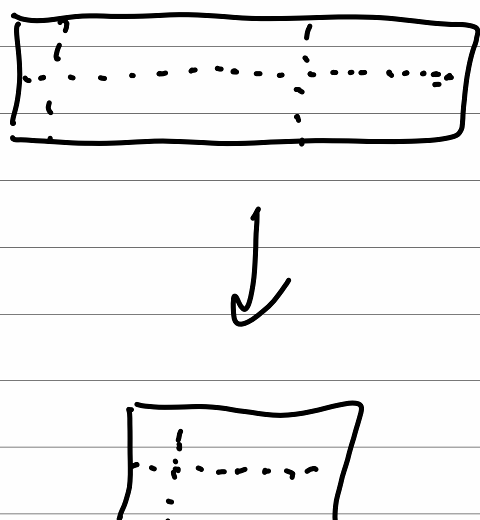
We have $h_*(\beta \alpha \beta^{-1}) = h_*(\alpha^{-1})$ and so

$$\alpha^r \beta^s \alpha^p \beta^q \beta^{-s} \alpha^{-r} = \alpha^{r-p-r} \beta^q = \alpha^{-p} \beta^q$$

must be equal to $h_*(\bar{\lambda}') = \beta^{-q}\gamma^{-p} = \gamma^{-p}\beta^{-q}$.

Hence $q = 0$.

Now, the loop ω_1 lifts to two loops in T^2 and ω_2^2 lifts to a loop meeting each of the lifts of ω_1 transversely at a single point.



In terms of group elements, a lift of w_1 gives element a^p, and the lift of $w_2^2 = \gamma^r \beta^q \gamma^r \beta^q = \beta^{2q}$ gives b^q.

By Lemma 2.7,

$$\begin{vmatrix} p & 0 \\ 0 & q \end{vmatrix} = pq = \pm 1. \qquad \square$$

<u>Corollary</u> There is only one non-orientable genus 1 splitting.

<u>Proof</u> V_1 is K with the loop α filled in.

Suppose that the Heegaard splitting is $(V_1; J)$. Since J has to be a ω_1 from the previous result, J represents a^p, $p = \pm 1$.

So M is obtained by attaching a disc to a, and then filling in the boundary with a 3-ball.

This clearly gives the non-orientable S^2-bundle over S^1. $\square$

3. Connected sums

Let M, M_1, M_2 be connected 3-manifolds. Let $B_i \subseteq \operatorname{Int} M_i$ be two 3-cells. Let $R_i = \overline{M_i \setminus B_i}$.

Suppose we have embeddings $h_i : R_i \hookrightarrow M$ such that $M = \operatorname{im} h_1 \cup \operatorname{im} h_2$, and $\operatorname{im} h_1 \cap \operatorname{im} h_2 = h_1(\partial B_1) = h_2(\partial B_2)$.

We say that M is the connected sum of M_1 and M_2, and we write $M = M_1 \# M_2$.

Starting with M_1 and M_2, it is irrelevant by Theorem 1.5 how we choose B_1 and B_2. By Theorem 1.4, there are precisely two ways of identifying ∂B_1 with ∂B_2.

If M, M_1, M_2 are oriented, the identification must be orientation reversing.

Hence we have:

<u>Lemma 3.1</u> Connected sum is a well-defined commutative associative

operation in the category of oriented 3-manifolds (with orientation preserving homeomorphisms as morphisms.

On the other hand, we will show that there exist oriented 3-manifolds M_1, M_2 with $M_1 \# M_2 \not\cong M_1 \# -M_2$, where $-M_2$ is M_2 with opposite orientation.

Despite this, we will use the terminology "connected sum", and address the possible ambiguity in the non-oriented case.

Note that for any M_1, M_2, there are at most two 3-manifolds which we could call $M_1 \# M_2$, and exactly one if M_i admits a self homeomorphisms which fixes a point and reverses the orientation of a n'hood of this point.

Definition

Given a 3-manifold M, we denote by $\hat{M}$ the 3-manifold obtained from M by capping each component of ∂M homeomorphic to S^2 with a 3-cell.

Thus $M \subseteq \hat{M}$, and the closure of every component of $\hat{M} \setminus M$ is a 3-cell, $\partial \hat{M}$ contains no 2-spheres, and the map $i_* : \pi_1(M) \longrightarrow \pi_1(\hat{M})$

induced by the inclusion $\iota: M \hookrightarrow \hat{M}$

is an isomorphism.

[using van Kampen's theorem].

The same theorem implies:

<u>Lemma 3.2</u>

If $M_1 \# M_2$ is a connected sum

decomposition of a 3-manifold M,

then there is a natural isomorphism

$$\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2).$$

Our goal is a uniqueness statement.